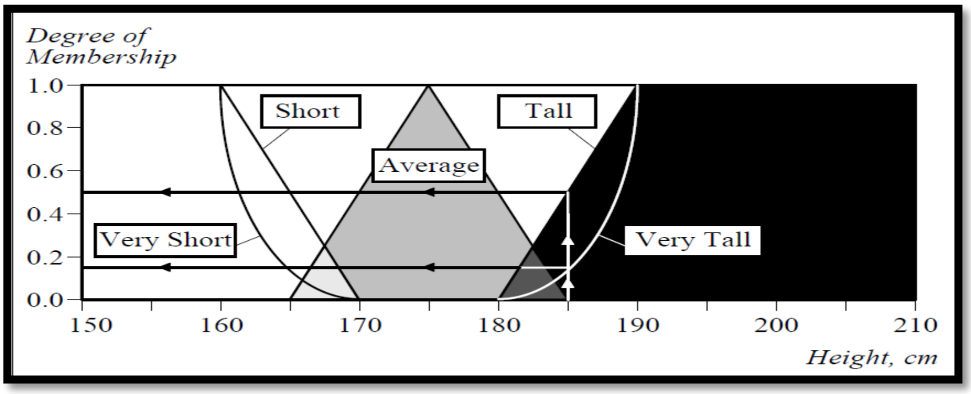
Neural Networks and Fuzzy Logic (630514)

Lecture 19

Operations and rules of fuzzy sets

Linguistic variables and hedges:

- **Linguistic Variables**: Linguistic variables are the **input or output** variables of the system whose values are words from a natural language, instead of numerical values. The fuzzy set theory is rooted in linguistic variables.
- A linguistic variable is a fuzzy variable. For example:
    - The statement "**John is tall**" implies that the ***linguistic variable*** **John** takes the ***linguistic Value*** **tall**.
- The range of possible values of a linguistic variable represents the **universe of discourse** of that variable.
    - For example, the universe of discourse of the linguistic variable **speed** might be the range between **0 and 220 km/h** and may include such fuzzy subsets as **very slow, slow, medium, fast, and very fast.**
- A linguistic variable carries with it the concept of **fuzzy set qualifiers**, called **hedges**.
- **Hedges** are terms that modify the shape of fuzzy sets. They include adverbs such as **very**, **somewhat**, **quite**, **more or less** and **slightly**.
- **Hedges** act as **operations** themselves, For instance.
    - **Very** performs **concentration** and ***creates a new subset***. From the set of tall men, it derives the subset of very tall men. **Extremely** serves the same purpose to a ***greater extent***.
    - **More or less** performs **dilation**; for example, the set of more or less tall men is broader than the set of tall men (dilation is an operation opposite to concentration. It expands the set).
    - **Indeed**, the operation of **intensification**, it can be done by **increasing** the degree of membership above **0.5** and **decreasing** those below **0.5.**
- **Hedges** are useful as **operations**, but they can also break down continuums into fuzzy intervals:
    - For example, the following hedges could be used to describe temperature: very cold, moderately cold, slightly cold, neutral, slightly hot, moderately hot and very hot.
- Hedges often used in practical applications:
    - **Very**, ***the operation of concentration***, it reduces the degree of membership of fuzzy elements using a mathematical square equation, for example: **0.86** membership in the set of tall men will be **0.7396** membership in the set of very tall men.

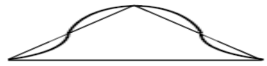


Fuzzy sets with the hedge very

- **Extremely**, ***same purpose as* very**, for example: **0.86** membership in the set of tall men will be **0.6361** membership in the set of extremely tall men.
- **Very very,** an extension of concentration, for example: **0.86** membership in the set of tall men will be **0.5470** in the set of very very tall men.
- **More or less**, the operation of **dilation** ***expands a set and increases the degree of membership***, for example: **0.86** membership in the set of tall men will be **0.9274** membership in the set of more or less tall men.
- **Indeed**, the operation of **intensification**, it can be done by **increasing** the degree of membership above **0.5** and **decreasing** those below **0.5.**For example: **0.86** membership in the set of tall men will be **0.9608** membership in the set of indeed tall men and **0.24** membership in tall men set, will have a **0.1152** membership in the indeed tall men set.
- In the next figure the **hedges**, **mathematical expression** and **graphical representation** are presented.

Representation of hedges in fuzzy logic:

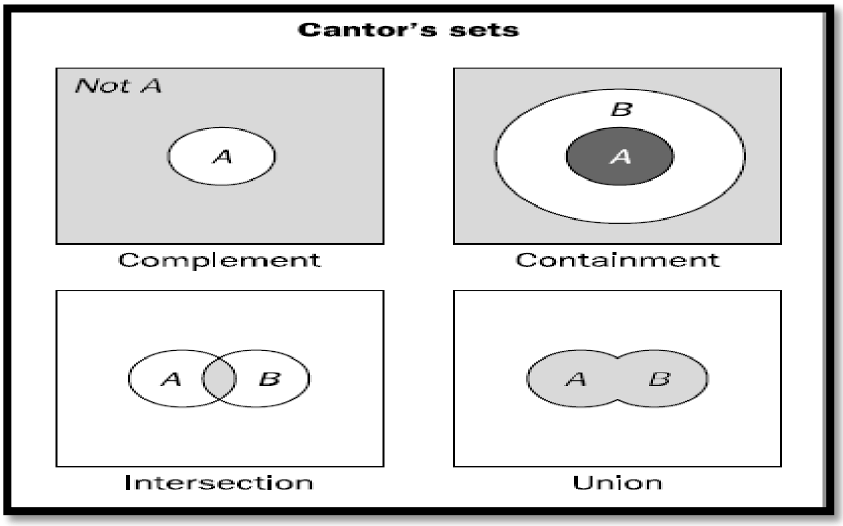
Hedge	*Mathematical Expression*	*Graphical Representation*
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^{2}$	
Extremely	$[\mu_A(x)]^{3}$	

Hedge	Mathematical Expression	Graphical Representation
Very very	$[\mu_A(x)]^4$	
More or less	$\sqrt{\mu_A(x)}$	
Somewhat	$\sqrt{\mu_A(x)}$	
Indeed	$2\,[\mu_A(x)]^2$ if $0 \le \mu_A \le 0.5$ $1 - 2\,[1 - \mu_A(x)]^2$ if $0.5 < \mu_A \le 1$	

Example: for Alex is tall, **μ = .86**, Alex is very tall, **μ = μ^2 =.74**, Alex is very very tall, **μ = μ^4 =.55.**

Operations of fuzzy sets:

- The classical set theory describes interactions between crisp sets. These interactions are called **operations**.
- **Cantor's sets:**



Operations on classical sets

1) Complement:

- **Crisp Sets**: ***Who does not belong to the set***?
- **Fuzzy Sets**: ***How much do elements not belong to the set?***
- It defined as the collection of all elements in the universe that do not reside in the set A.

$$\bar{A} = \{x / x \notin A, x \in X\}.$$

- The complement of a set is an opposite of this set. For example, if we have the set of tall men, its complement is the set of NOT tall men. If **A** is the fuzzy set, its complement **¬A** can be found as follows:

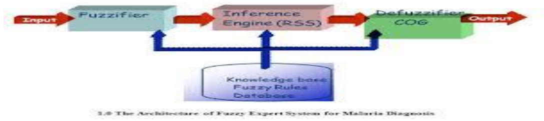
$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

- **Example**: given a fuzzy set of tall men

  Tall men **(0/180, 0.25/182.5, 0.5/185, 0.75/187.5, 1/190)**

  - the fuzzy set of NOT tall men will be:

  NOT tall men **(1/180, 0.75/182.5, 0.5/185, 0.25/187.5, 0/190)**



2) Containment:

- **Crisp Sets**: ***Which sets belong to which other sets?***
- **Fuzzy Sets**: ***Which sets belong to other sets?***
- A set can contain other sets. The smaller set is called the **subset**.
    - For example, the set of **tall men** contains all tall men; **very tall men** is a subset of **tall men**. However, the **tall men** set is just a subset of the set of **men**.
    - **In crisp sets**, all elements of a subset entirely belong to a larger set.
    - **In fuzzy sets**, however, each element can belong less to the subset than to the larger set.
    - The two sets **A** and **B** in universe, if one set (**A**) is contained in another set B, then

$$A \subseteq B \rightarrow \mu_A(x) \leq \mu_B(x)$$

- **Example:**

  Tall men (0/180, 0.25/182.5, 0.50/185, 0.75/187.5, 1/190)

  Very tall men (0/180, 0.06/182.5, 0.25/185, 0.56/187.5, 1/190)

3) Intersection:

- **Crisp Sets**: ***Which element belongs to both sets?***
- **Fuzzy Sets: *To what degree the element is in both sets?***
- **In classical set** theory, an intersection between two sets contains the elements shared by these sets.
    - For example, the intersection of the set of **tall men** and the set of **fat men** is the area where these sets overlap.
    - It represents all those elements in the universe **X** that simultaneously reside in (or belongs to) both sets **A** and **B**.

$$A \cap B = \{x/x \in A \text{ and } x \in B\}.$$

- **In fuzzy sets**, an element may partly belong to both sets with different memberships.
    - A fuzzy intersection is the lower membership in both sets of each element. The fuzzy intersection of two fuzzy sets **A** and **B** on universe of discourse **X**:

$$\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)] = \mu_A(x) \cap \mu_B(x), \text{ where } x \in X$$

- **Example:**

Tall men (0/165, 0/175, 0.0/180, 0.25/182.5, 0.5/185, 1/190)

Average men (0/165, 1/175, 0.5/180, 0.25/182.5, 0.0/185, 0/190)

The intersection of these two sets is:

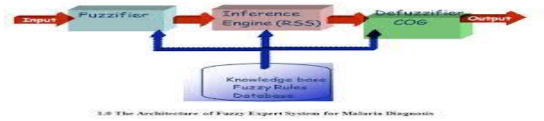
Tall men ∩ average men (0/165, 0/175, 0/180, 0.25/182.5, 0/185, 0/190)

Or

Tall men ∩ average men (0/180, 0.25/182.5,0/185)

4) Union

- **Crisp Sets: *Which element belongs to either set?***
- **Fuzzy Sets: *How much of the element is in either set?***



- **In classical set**, **Union** represents all the elements in the universe that reside in either the set **A**, the set **B** or both sets **A** and **B**. This operation is called the **logical OR**.

  A ∪ B = {x/x ∈ A or x ∈ B}.

- For example, the union of **tall men** and **fat men** contains all men who are **tall OR fat**.

- **In fuzzy sets**, the union is the reverse of the intersection. That is, the union is the **largest** membership value of the element in either set.
  - The fuzzy operation for forming the union of two fuzzy sets **A** and **B** on universe **X** can be given as:

$$\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)] = \mu_A(x) \cup \mu_B(x), \text{ where } x \in X$$

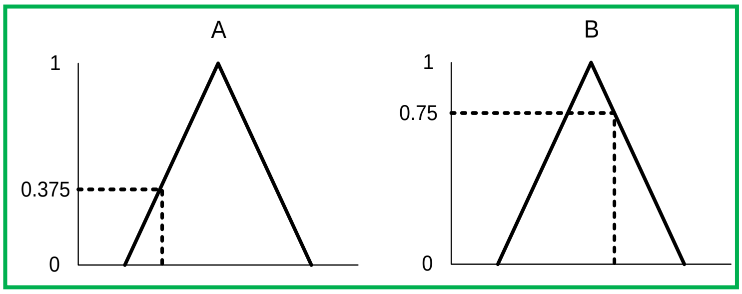
- **Example:**

Tall men	**(0/165, 0/175, 0.0/180, 0.25/182.5, 0.5/185, 1/190)**
Average men	**(0/165, 1/175, 0.5/180, 0.25/182.5, 0.0/185, 0/190)**

The union of these two sets is:

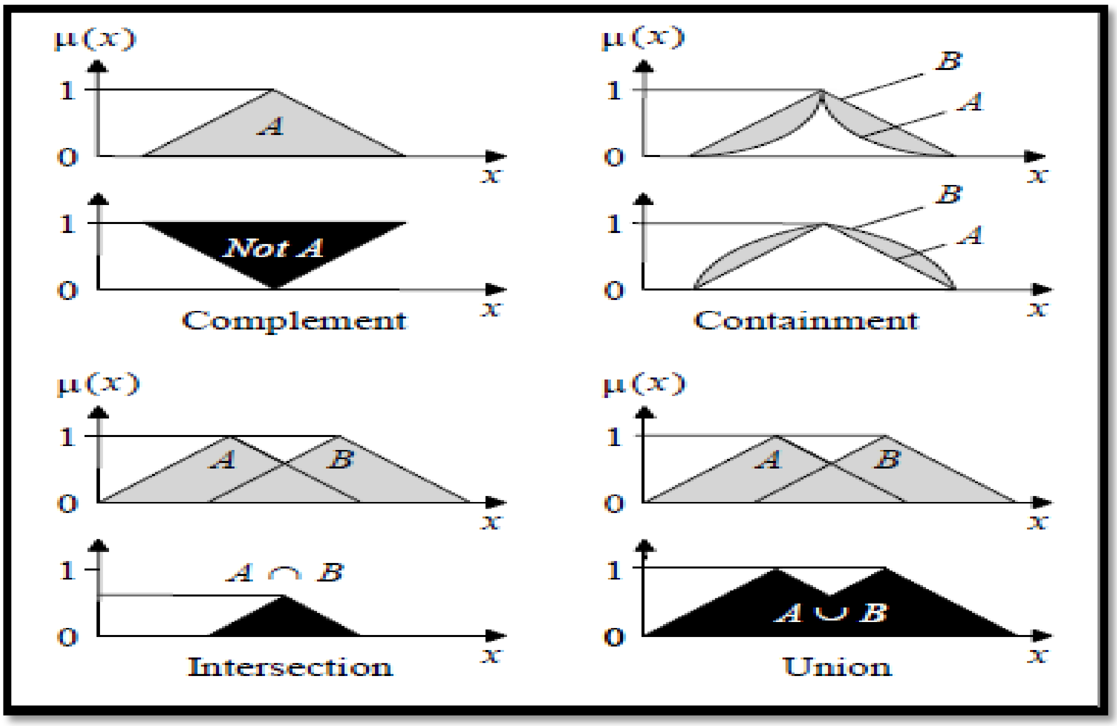
Tall men ∪ **average men** **(0/165,1/175,0.5/180, 0.25/182.5, 0.5/185, 1/190)**

- **Example:**

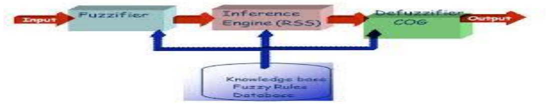


(A∨B = C) ⇒ (C = 0.75)

(A∧B = C) ⇒ (C = 0.375)



Operations of fuzzy sets



Operations on fuzzy sets:

- ***Crisp and fuzzy sets*** have the ***same properties***; frequently used properties of fuzzy sets are described below.

Consider two sets **A** and **B** defined on the universe **X**.

1) **Commutativity:**

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- **Example:**

  Tall men OR short men = short men OR tall men
  Tall men AND short men = short men AND tall men

2) **Associativity:**

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- **Example:**
  - **Tall men OR (short men OR average men) = (tall men OR short men) OR average men**
  - **Tall men AND (short men AND average men) = (tall men AND short men) AND average men**

3) **Distributivity:**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = A \cap B) \cup (A \cap C)$$

4) **Idempotency:**

$$A \cup A = A$$
$$A \cap A = A$$

5) **Identity:**

$$A \cup \emptyset = A$$
$$A \cap X = A$$
$$A \cap \emptyset = \emptyset$$
$$A \cup X = X.$$

- **Example:**
  - **tall men OR undefined = tall men**
  - **tall men AND unknown = tall men**
  - **tall men AND undefined = undefined**
  - **tall men OR unknown = unknown**

  Where
  - **Undefined** is an empty (**null**) set, the set having all degree of memberships equal to **0**.
  - **Unknown** is a set having all degree of memberships equal to **1**.

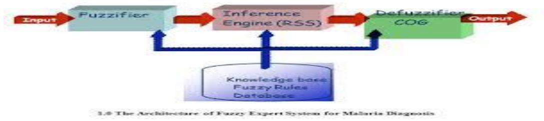
6) **Transitivity:**

If $A \subseteq B \subseteq C$, then $A \subseteq C$, $\subseteq$ means contained.

- Every set contains the subsets of its subsets.

- **Example:**

  IF (extremely tall men ⊆ very tall men)
  AND (very tall men ⊆ tall men)
  THEN (extremely tall men ⊆ tall men)



7) Involution:

$$\neg(\neg A) = A$$

8) De Morgan's Law:

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$
$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

- Using fuzzy set operations, their properties and hedges, we can easily obtain a variety of fuzzy sets from the existing ones.
  - **For example**, if we have fuzzy set **A** of tall men and fuzzy set **B** of short men, we can derive fuzzy sets:
    - **C** of not very tall men and not very short men:

$$\mu_C(x) = [1 - \mu_A(x)^2] \cap [1 - (\mu_B(x)^2]$$

    - **D** of not very very tall and not very very short men:

$$\mu_D(x) = [1 - \mu_A(x)^4] \cap [1 - (\mu_B(x)^4]$$

Solved Example:

Given the classical sets,

A = {9, 5, 6, 8, 10}

B = {1, 2, 3, 7, 9}

C = {1, 0}

Defined on universe

X = {Set of all 'n' natural no}

Prove the classical set properties associativity.

Solution: The associative property is given by

1. A ∪ (B ∪ C) = (A ∪ B) ∪ C.

LHS

A ∪ (B ∪ C)

(a) **B ∪ C = {2, 3, 7, 9, 1, 0}.**

(b) **A ∪ (B ∪ C) = {5, 6, 8, 10, 2, 3, 7, 9, 1, 0}.**

RHS

(A ∪ B) ∪ C

(a) **(A ∪ B) = {9, 5, 6, 8, 10, 1, 2, 3, 7}.**

(b) **(A ∪ B) ∪ C = {9, 5, 6, 10, 8, 1, 2, 3, 7, 0}.**

LHS = RHS

A ∪ (B ∪ C) = (A ∪ B) ∪ C.

2. (A ∩ (B ∩ C) = (A ∩ B) ∩ C

LHS

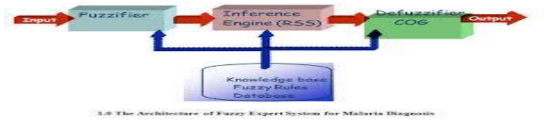
(a) **(B ∩ C) = {1}.**

(b) **A ∩ (B ∩ C) = {∅}.**

RHS

(A ∩ B) ∩ C

(a) **(A ∩ B) = {9}.**



(b) (A ∩ B) ∩ C = {∅}.

LHS = RHS

A ∩ (B ∩ C) = (A ∩ B) ∩ C.

Fuzzy rules:

- A **fuzzy rule** can be defined as a conditional statement in the form:

  IF x is A THEN y is B

  Where **x** and **y** are **linguistic variables**; and **A** and **B** are **linguistic values**.
- **Antecedent** - The **initial** (or "**if**") part of a fuzzy rule.
- **Consequent** - The **final** (or "**then**") part of a fuzzy rule.
- An example might be a **temperature controller**:

  If the break temperature is HOT
  and the speed is NOT VERY FAST
  then break SLIGHTLY DECREASED.

What is the difference between classical and fuzzy rules?

- A classical **IF-THEN** rule uses binary logic, for example,

Rule: 1

IF speed is > 100
THEN stopping_distance is long

Rule: 2

IF speed is < 40
THEN stopping_distance is short

- The variable **speed** can have any numerical value between **0** and **220 km/h**, but the linguistic variable **stopping_distance** can take either value **long** or **short**.
- In other words, classical rules are expressed in the **black-and-white** language of Boolean logic.
- We can also represent the stopping distance rules in a **fuzzy form**:

Rule: 1

IF speed is fast
THEN stopping_distance is long

Rule: 2

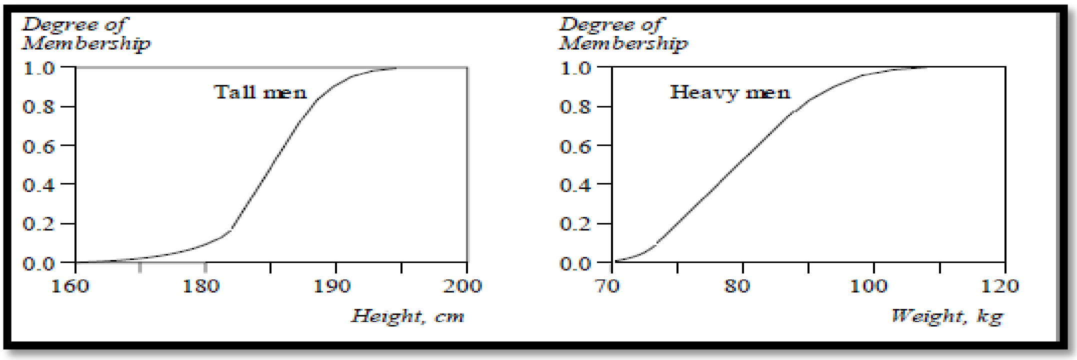
IF speed is slow
THEN stopping_distance is short

- In **fuzzy rules**, the linguistic variable **speed** also has the range (the universe of discourse) between **0** and **220 km/h**, but this range includes fuzzy sets, such as **slow, medium** and **fast**. The universe of discourse of the linguistic variable **stopping_distance** can be between **0** and **300 m** and may include such fuzzy sets as **short, medium** and **long**.

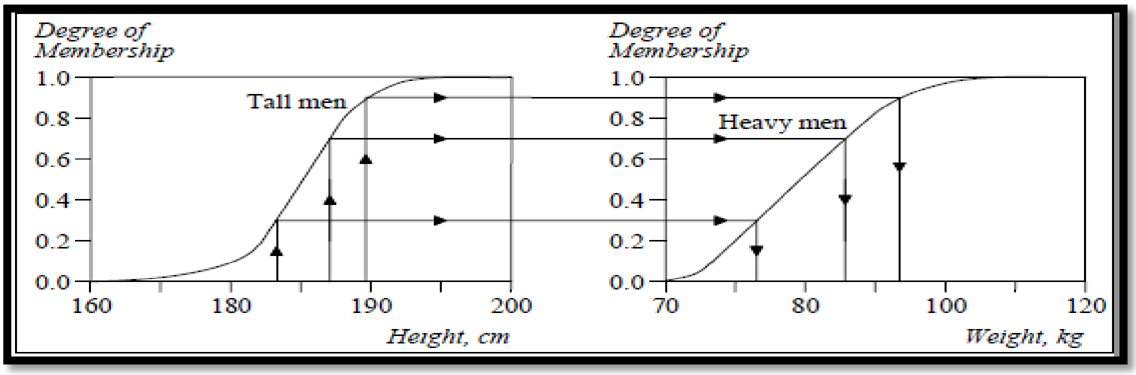
Fuzzy rules relate fuzzy sets:

- In a **fuzzy system**, **all rules fire to some extent**, i.e., fire partially. If the antecedent is true to some degree of membership, then the consequent is also true to that same degree.

Example: Fuzzy sets of tall and heavy men.



- These fuzzy sets provide the basis for a weight estimation model. The model is based on a relationship between a man's height and his weight:
  IF height is tall THEN weight is heavy
- The value of the output or a truth membership grade of the rule consequent **can be estimated directly from** a corresponding truth membership grade in the antecedent. This form of fuzzy inference uses a method called **monotonic selection**.



- A fuzzy rule can have **multiple antecedents**, for example:
  - **IF** **project_duration is long**
    AND **project_staffing is large**
    AND **project_funding is inadequate**
    THEN **risk is high.**
  - **IF** **service is excellent**
    OR **food is delicious**
    THEN **tip is generous.**
- The **consequent** of a fuzzy rule can also include **multiple parts**, for instance:
  IF **temperature is hot**
  THEN **hot_water is reduced;**
  cold_water is increased
- **How are all these output fuzzy sets combined and transformed into a single number?**
  - To obtain a **single crisp solution**, a fuzzy expert system first **aggregates** all output fuzzy sets into a single output fuzzy set, and then **defuzzifies** the resulting fuzzy set into a **single number**.